

New extended superconformal sigma models and Quaternion Kähler manifolds

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Abstract

Quaternion Kähler manifolds are known to be the target spaces for matter hypermultiplets coupled to $\mathcal{N} = 2$ supergravity. It is also known that there is a one-to-one correspondence between $4n$ -dimensional quaternion Kähler manifolds and those $4(n+1)$ -dimensional hyperkähler spaces which are the target spaces for rigid superconformal hypermultiplets (such spaces are called hyperkähler cones). In this paper we present a projective-superspace construction to generate a hyperkähler cone $\mathcal{M}_H^{4(n+1)}$ of dimension $4(n+1)$ from a $2n$ -dimensional real analytic Kähler-Hodge manifold \mathcal{M}_K^{2n} . The latter emerges as a maximal Kähler submanifold of the $4n$ -dimensional quaternion Kähler space \mathcal{M}_Q^{4n} such that its Swann bundle coincides with $\mathcal{M}_H^{4(n+1)}$. Our approach should be useful for the explicit construction of new quaternion Kähler metrics. The results obtained are also of interest, e.g., in the context of supergravity reduction $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$, or alternatively from the point of view of embedding $\mathcal{N} = 1$ matter-coupled supergravity into an $\mathcal{N} = 2$ theory.

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1 Introduction

Many years ago, Bagger and Witten [1] demonstrated that the scalar fields of matter hypermultiplets coupled to 4D $\mathcal{N} = 2$ supergravity take their values in a $4n_{\text{H}}$ -dimensional quaternion Kähler manifold $\mathcal{M}_Q^{4n_{\text{H}}}$, unlike the rigid supersymmetric case where the hypermultiplet target spaces are hyperkähler [2]. It was also pointed out in [1] that the problem of reduction from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supergravity is nontrivial. For such a reduction, it is not sufficient to simply switch off one of the two gravitinos as well as the graviphoton. In addition, it is also necessary to restrict the scalar fields to lie in a $2n_{\text{H}}$ -dimensional

Kähler-Hodge¹ submanifold $\mathcal{M}_K^{2n_H}$ of the $4n_H$ -dimensional quaternion Kähler space $\mathcal{M}_Q^{4n_H}$. Provided a required Kähler-Hodge submanifold $\mathcal{M}_K^{2n_H}$ of $\mathcal{M}_Q^{4n_H}$ exists and is constructed *explicitly*, the supergravity reduction $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ has been worked out by Andrianopoli, D’Auria and Ferrara [5], building in part on the mathematical results of [6]. On the other hand, if one is interested in embedding $\mathcal{N} = 1$ matter-coupled supergravity into an $\mathcal{N} = 2$ theory, one has to ask two different questions that can be formulated as follows. First, given a $2n_H$ -dimensional Kähler-Hodge manifold $\mathcal{M}_K^{2n_H}$, does there exist a quaternion Kähler manifold $\mathcal{M}_Q^{4n_H}$ such that $\mathcal{M}_K^{2n_H}$ is its submanifold? Second, if the answer to the first question is “Yes,” can one develop a regular procedure to generate $\mathcal{M}_Q^{4n_H}$ starting from $\mathcal{M}_K^{2n_H}$? In this paper, we will argue that a natural formalism to address these questions is the concept of rigid projective superspace [7, 8] (see also [9] for a review) and its extension to the case of supergravity with eight supercharges elaborated in [10, 11].

It is known that the study of quaternion Kähler manifolds is related to that of hyperkähler spaces with special properties. More precisely, there exists a one-to-one correspondence [12] (see also [13]) between $4n$ -dimensional quaternion Kähler manifolds and $4(n+1)$ -dimensional hyperkähler spaces possessing a homothetic Killing vector, and hence an isometric action of $SU(2)$ rotating the complex structures. Such hyperkähler spaces, known in the mathematics literature as “Swann spaces” and often referred to as “hyperkähler cones” in the physics literature, are the target spaces for rigid $\mathcal{N} = 2$ superconformal sigma models [14, 15]. The above correspondence is natural from the point of view of the $\mathcal{N} = 2$ superconformal tensor calculus [16], or more generally within the harmonic-superspace [17, 18] and the projective-superspace [11] approaches to four-dimensional $\mathcal{N} = 2$ matter-coupled supergravity. In the context of $\mathcal{N} = 2$ supersymmetric sigma models, the quaternion Kähler manifold $\mathcal{M}_Q^{4n_H}$ associated to a $4(n+1)$ -dimensional hyperkähler cone $\mathcal{M}_H^{4(n_H+1)}$ is obtained by applying the procedure elaborated in some detail in [15] and later on applied in many publications, see, e.g., [19, 20] for an incomplete list.

In the present paper, we concentrate on deriving new hyperkähler cones with interesting geometric properties. *We give a new method for finding hyperkähler cones and thus quaternion Kähler manifolds, and also demonstrate the surprising existence of a maximal Kähler submanifold $\mathcal{M}_K^{2n_H}$ of the quaternion Kähler manifold $\mathcal{M}_Q^{4n_H}$.*

In the curved projective-superspace setting, general hypermultiplet matter couplings to $\mathcal{N} = 2$ supergravity were presented in [11] and [21]. The two families of locally supersymmetric sigma models introduced in [11] and [21] are dual to each other. They

¹This type of geometry corresponds to nonlinear couplings in $\mathcal{N} = 1$ supergravity [3, 4].

involve the same matter hypermultiplets, which are described in terms of n_{H} covariant weight-zero arctic multiplets Υ^I and their smile-conjugates² $\check{\Upsilon}^{\bar{I}}$, but differ in their (second) conformal compensators used.³ In the case of the model of [11], the compensators are a covariant weight-one arctic multiplet Ξ and its smile-conjugate $\check{\Xi}$. The compensator in the model of [21] is a covariant tensor multiplet H . In both models, the matter $\mathcal{N} = 2$ superfields Υ^I and $\check{\Upsilon}^{\bar{I}}$ take their values in a Kähler manifold $\mathcal{M}_K^{2n_{\text{H}}}$ with the Kähler potential $K(\Phi^I, \bar{\Phi}^{\bar{J}})$. Our goal in this paper is to study rigid superconformal versions of the models in [11, 21] which are obtained by retaining intact the compensator(s) but replacing the supergravity covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\alpha}i})$ with those corresponding to a conformally flat superspace. Technically the rigid superconformal version of the sigma model in [21] is simpler to deal with, for the tensor compensator H is shorter⁴ than the arctic one, Ξ . That is why we will concentrate on the study of this model. Some aspects of the superconformal sigma model derived from [11] were studied in [25] where that model was first introduced.

This paper is organized as follows. In section 2 we introduce the off-shell $\mathcal{N} = 2$ superconformal sigma model to be studied and discuss its geometric aspects. The main thrust of section 3 is to argue that the off-shell $\mathcal{N} = 2$ superconformal symmetry of the model can be used to convert the infinite set of algebraic auxiliary field equations into a single second-order differential equation under given initial conditions, which is a deformation of the geodesic equation, with the complex coordinate for \mathbb{CP}^1 being the evolution parameter. In section 4 we explicitly eliminate the auxiliary superfields and derive the hypermultiplet Lagrangian in terms of the physical superfields, in the case when $\mathcal{M}_K^{2n_{\text{H}}}$ is chosen to be $\mathbb{CP}^{n_{\text{H}}}$. Section 5 is devoted to the discussion of the results obtained. Two technical appendices are also included. In Appendix A we list the $\mathcal{N} = 2$ superconformal transformations of several off-shell supermultiplets and their realization in $\mathcal{N} = 1$ superspace. Appendix B contains a few results concerning the $\mathcal{N} = 2$ supersymmetric sigma models on (co)tangent bundles of Hermitian symmetric spaces.

²An arctic multiplet Υ and its smile-conjugate $\check{\Upsilon}$ form a polar multiplet, according to the terminology introduced in [22].

³As discussed in [21], the sigma-model couplings of [11] and [21] are $\mathcal{N} = 2$ analogues of the well-known matter couplings in the old minimal and the new minimal formulations for $\mathcal{N} = 1$ supergravity, see [23, 24] for reviews.

⁴When realized in terms of $\mathcal{N} = 1$ superfields, the arctic multiplet [8] includes two physical superfields (one chiral and one complex linear) and an infinite number of auxiliary superfields, see section 2 for more detail. On the contrary, the tensor multiplet consists of two physical superfields only.

2 The sigma model and its geometric properties

In this paper we are interested in a rigid superconformal version of the four-dimensional $\mathcal{N} = 2$ locally supersymmetric model proposed in [21]. This theory is formulated in $\mathcal{N} = 2$ projective superspace [8], and therefore its action can naturally be written either in terms of $\mathcal{N} = 2$ projective superfields or in terms of the associated $\mathcal{N} = 1$ superfields. We will use both realizations in different parts of this paper, and the latter will be used to formulate the action. It consists of two terms,

$$S[H(\zeta), \Upsilon(\zeta)] = \kappa S_{\text{T}} + S_{\text{H}}, \quad (2.1)$$

where κ is a constant parameter, and

$$S_{\text{T}} = - \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta H \ln H, \quad (2.2)$$

$$S_{\text{H}} = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta H K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}), \quad (2.3)$$

with some closed integration contours in the ζ -plane. Here $H(\zeta)$ is an $\mathcal{O}(2)$ multiplet [7] (or $\mathcal{N} = 2$ tensor multiplet [26])

$$H(\zeta) = \frac{1}{\zeta} \varphi + G - \zeta \bar{\varphi}, \quad \bar{D}_{\dot{\alpha}} \varphi = 0, \quad \bar{D}^2 G = 0, \quad \bar{G} = G, \quad (2.4)$$

$\Upsilon^I(\zeta)$ a set of arctic hypermultiplets [8], $I = 1, \dots, n_{\text{H}}$,

$$\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n^I = \Phi^I + \zeta \Sigma^I + \mathcal{O}(\zeta^2), \quad \bar{D}_{\dot{\alpha}} \Phi^I = 0, \quad \bar{D}^2 \Sigma = 0, \quad (2.5)$$

and $\check{\Upsilon}^{\bar{I}}(\zeta)$ their smile-conjugates

$$\check{\Upsilon}^{\bar{I}}(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \bar{\Upsilon}_n^{\bar{I}}. \quad (2.6)$$

The $\mathcal{N} = 1$ superfields $\Upsilon_2^I, \Upsilon_3^I, \dots$, are complex unconstrained. Since these appear in the action without derivatives, they are purely auxiliary degrees of freedom. The hypermultiplet action S_{H} involves the Kähler potential, $K(\Phi^I, \bar{\Phi}^{\bar{J}})$, of a real-analytic Kähler manifold $\mathcal{M} \equiv \mathcal{M}_K^{2n_{\text{H}}}$ of complex dimension n_{H} .

The action S_{T} is the $\mathcal{N} = 2$ projective-superspace formulation [7] of the $\mathcal{N} = 2$ improved tensor multiplet model [27]. Its realization in terms of $\mathcal{N} = 1$ superfields was first developed in [28]:

$$S_{\text{T}} = \int d^4x d^4\theta L_{\text{T}}(G, \varphi, \bar{\varphi}), \quad L_{\text{T}}(G, \varphi, \bar{\varphi}) = \mathbb{H} - G \ln(G + \mathbb{H}), \quad (2.7)$$

where

$$\mathbb{H} := \sqrt{G^2 + 4\varphi\bar{\varphi}} . \quad (2.8)$$

The combination $G + \mathbb{H}$ naturally originates, e.g., as follows:

$$H(\zeta) = \frac{1}{2}(G + \mathbb{H}) \left(1 - \zeta \frac{2\bar{\varphi}}{G + \mathbb{H}}\right) \left(1 + \frac{1}{\zeta} \frac{2\varphi}{G + \mathbb{H}}\right) . \quad (2.9)$$

The theory with action (2.1) is $\mathcal{N} = 2$ superconformal provided $H(\zeta)$ transforms as a $\mathcal{N} = 2$ superconformal tensor multiplet and Υ^I as a superconformal weight-zero arctic multiplet [25]. The corresponding transformation laws are given below in eqs. (3.5a) and (3.5b) respectively.

As discussed in [21], the theory (2.1) possesses a dual formulation obtained by dualizing the tensor multiplet $H(\zeta)$ into an arctic multiplet $\Xi(\zeta)$ and its conjugate following the procedure given, e.g., in [22]. The resulting hypermultiplet sigma model [25]

$$S_{\text{dual}}[\Xi(\zeta), \Upsilon(\zeta)] = \kappa \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \check{\Xi} \Xi \exp \left\{ \frac{1}{\kappa} K(\Upsilon, \check{\Upsilon}) \right\} \quad (2.10)$$

is $\mathcal{N} = 2$ superconformal provided Ξ transforms as the superconformal weight-one arctic multiplet, see Appendix A for the corresponding transformation law. The above theory is the rigid superconformal limit of the locally supersymmetric sigma model proposed in [11]. On the other hand, the theory with action (2.1) is the rigid superconformal version of the locally supersymmetric sigma model proposed in [21].

As pointed out in [21], the theory (2.1) is a natural extension of the $\mathcal{N} = 1$ superconformal sigma model:

$$S[G, \Phi] = -\kappa \int d^4x d^4\theta G \ln G + \int d^4x d^4\theta G K(\Phi^I, \bar{\Phi}^{\bar{J}}) . \quad (2.11)$$

Here the first term is proportional to the action for the $\mathcal{N} = 1$ improved tensor multiplet [29]. The dual version of (2.11) is

$$S_{\text{dual}}[\chi, \Phi] = k \int d^4x d^4\theta \bar{\chi} \chi \exp \left\{ \frac{1}{\kappa} K(\Phi, \bar{\Phi}) \right\} , \quad (2.12)$$

with χ a chiral scalar superfield. As is known, the action $S_{\text{dual}}[\chi, \Phi]$ is obtained from that describing chiral matter in $\mathcal{N} = 1$ supergravity (see, e.g., [23, 24] for reviews) by switching off the (axial) vector gravitational superfield and keeping intact the chiral compensator χ and its conjugate. Clearly, the superconformal sigma model (2.10) is an $\mathcal{N} = 2$ extension of (2.12).

The extended superconformal sigma model (2.1) inherits all the geometric features of its $\mathcal{N} = 1$ predecessor (2.11). The Kähler invariance of the latter,

$$K(\Phi, \bar{\Phi}) \longrightarrow K(\Phi, \bar{\Phi}) + F(\Phi) + \bar{F}(\bar{\Phi}) \quad (2.13)$$

turns into

$$K(\Upsilon, \check{\Upsilon}) \longrightarrow K(\Upsilon, \check{\Upsilon}) + F(\Upsilon) + \bar{F}(\check{\Upsilon}) \quad (2.14)$$

for the model (2.1), where we have used the identity

$$\oint \frac{d\zeta}{\zeta} \int d^4x d^4\theta H F(\Upsilon) = 0 , \quad (2.15)$$

for any holomorphic function $F(\Phi)$. A holomorphic reparametrization of the Kähler manifold \mathcal{M} ,

$$\Phi^I \longrightarrow f^I(\Phi) , \quad (2.16)$$

has the following counterpart

$$\Upsilon^I(\zeta) \longrightarrow f^I(\Upsilon(\zeta)) \quad (2.17)$$

in the $\mathcal{N} = 2$ case. Therefore, the physical $\mathcal{N} = 1$ superfields of the $\mathcal{N} = 2$ arctic multiplet

$$\Upsilon^I(\zeta) \Big|_{\zeta=0} = \Phi^I , \quad \frac{d\Upsilon^I(\zeta)}{d\zeta} \Big|_{\zeta=0} = \Sigma^I , \quad (2.18)$$

should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point. Thus the variables (Φ^I, Σ^J) parametrize the holomorphic tangent bundle $T\mathcal{M}$ of the Kähler manifold \mathcal{M} . This interpretation of the physical variables of the hypermultiplet theory (2.3) coincides with that proposed in [30] for the *non-superconformal* sigma model

$$S[\Upsilon(\zeta)] = \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}) . \quad (2.19)$$

which is obtained from (2.1) by “freezing” the tensor multiplet, that is by replacing $H(\zeta)$ with its ζ -independent expectation value $\langle H \rangle = \text{const.}$

Suppose that in the action (2.3) we have eliminated all the auxiliary superfields contained in Υ and $\check{\Upsilon}$ with the aid of the corresponding algebraic equations of motion

$$\oint \frac{d\zeta}{\zeta} \zeta^n H \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{d\zeta}{\zeta} \zeta^{-n} H \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \check{\Upsilon}^{\bar{I}}} = 0 . \quad n \geq 2 \quad (2.20)$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ denote their unique solution subject to the initial conditions (2.18)

$$\Upsilon_*(0) = \Phi, \quad \dot{\Upsilon}_*(0) = \Sigma. \quad (2.21)$$

The action (2.1) then turns into

$$\begin{aligned} S[G, \varphi, \Phi, \Sigma] &:= S[H(\zeta), \Upsilon_*(\zeta)] = \int d^4x d^4\theta L(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}), \\ L(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) &= \kappa L_T(G, \varphi, \bar{\varphi}) + L_H(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}). \end{aligned} \quad (2.22)$$

Here the tensor multiplet Lagrangian is given by eq. (2.7). In accordance with the generalized Legendre transform procedure [8], we should dualize the real linear superfield G into a chiral scalar χ and its conjugate $\bar{\chi}$, and further dualize the complex linear tangent variables Σ^I and their conjugates $\bar{\Sigma}^{\bar{I}}$ into chiral one-forms Ψ_I and their conjugates $\bar{\Psi}_{\bar{I}}$, $\bar{D}_{\dot{\alpha}}\chi = \bar{D}_{\dot{\alpha}}\Psi_I = 0$. This results in

$$S[G, \varphi, \Phi, \Sigma] \longrightarrow S[\chi, \varphi, \Phi, \Psi] = \int d^4x d^4\theta \mathcal{H}(\chi, \bar{\chi}, \varphi, \bar{\varphi}, \Phi^I, \bar{\Phi}^{\bar{J}}, \Psi_I, \bar{\Psi}_{\bar{J}}). \quad (2.23)$$

We thus have the following striking situation: The target space of this sigma model is a hyperkähler manifold, $HKC^{4(n_H+1)}(\mathcal{M})$, of real dimension $4n_H + 4$. Since the sigma model is $\mathcal{N} = 2$ superconformal, $HKC^{4(n_H+1)}(\mathcal{M})$ is a hyperkähler cone in the sense of [14, 15]. The Lagrangian \mathcal{H} in (2.23) is the hyperkähler potential for $HKC^{4(n_H+1)}(\mathcal{M})$. Note that the variables (Φ^I, Ψ_J) parametrize the holomorphic cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold $\mathcal{M} = \mathcal{M}_K^{2n_H}$. As mentioned in the introduction, there exists a one-to-one correspondence between $4n$ -dimensional quaternion Kähler spaces (QK) and $4(n+1)$ -dimensional hyperkähler cones (HKC) [12, 13]. In our case $HKC^{4(n_H+1)}(\mathcal{M}) \longleftrightarrow QK^{4n_H}(\mathcal{M})$. The Kähler manifold $\mathcal{M}_K^{2n_H}$ is embedded into $QK^{4n_H}(\mathcal{M})$.

3 Superconformal invariance and the auxiliary field equations

When dealing with the $\mathcal{N} = 2$ off-shell superconformal sigma-model (2.1), the main technical challenge is to explicitly eliminate the auxiliary superfields $\Upsilon_2^I, \Upsilon_3^I, \dots$, by means of solving the corresponding equations of motion (2.20). This section is devoted to a general analysis of the problem.

3.1 Superconformal invariance

Both actions (2.2) and (2.3) are $\mathcal{N} = 2$ superconformal. To write down the superconformal transformations of $H(\zeta)$, $\Upsilon(\zeta)$ and $\check{\Upsilon}(\zeta)$, it is useful to lift these multiplets to $\mathcal{N} = 2$ superspace $\mathbb{R}^{4|8}$ parametrized by coordinates $z^A = (x^a, \theta_i^\alpha, \bar{\theta}_{\dot{\alpha}}^i)$, where $i = \underline{1}, \underline{2}$. In the $\mathcal{N} = 2$ setting, each of $H(\zeta)$, $\Upsilon(\zeta)$ and $\check{\Upsilon}(\zeta)$ is a projective multiplet. In general, with respect to the $\mathcal{N} = 2$ Poincaré supersymmetry, a projective multiplet $Q(\zeta)$ is determined by the two conditions [8]:

(i) it is characterized by two fixed integers p, q (of which p may be equal to $-\infty$ and q to $+\infty$) such that

$$Q(z, \zeta) = \sum_p^q Q_n(z) \zeta^n ; \quad (3.1)$$

(ii) it is subject to the constraints

$$D_\alpha(\zeta)Q(\zeta) = \bar{D}_{\dot{\alpha}}(\zeta)Q(\zeta) = 0 , \quad (3.2)$$

where

$$D_\alpha(\zeta) := \zeta^i D_{\alpha i} , \quad \bar{D}_{\dot{\alpha}} := \zeta^i \bar{D}_{\dot{\alpha} i} , \quad \zeta^i := (1, \zeta) , \quad (3.3)$$

where $D_A = (\partial_a, D_\alpha^i, \bar{D}_{\dot{\alpha}}^i)$ are the $\mathcal{N} = 2$ flat covariant derivatives. With respect to the $\mathcal{N} = 2$ superconformal group, the admissible transformation laws prove to depend on the parameters p and q in (3.1) as shown in [25].

The following remark is needed here. It follows from the constraints (3.2) that the dependence of $Q(x, \theta_i, \bar{\theta}^i, \zeta)$ on the Grassmann variables $\theta_{\underline{2}}^\alpha$ and $\bar{\theta}_{\dot{\alpha}}^{\underline{2}}$ is uniquely determined in terms of its dependence on $\theta_{\underline{1}}^\alpha \equiv \theta^\alpha$ and $\bar{\theta}_{\dot{\alpha}}^{\underline{1}} \equiv \bar{\theta}_{\dot{\alpha}}$. In other words, the projective superfields depend effectively on half the Grassmann variables which can be chosen to be the spinor coordinates of $\mathcal{N} = 1$ superspace. In other words, no information is lost if we replace $Q(\zeta)$ by its $\mathcal{N} = 1$ projection $Q(\zeta)|$ defined as

$$U| = U(x, \theta_i, \bar{\theta}^i) \Big|_{\theta_{\underline{2}} = \bar{\theta}_{\dot{\alpha}}^{\underline{2}} = 0} , \quad (3.4)$$

for any $\mathcal{N} = 2$ superfield $U(x, \theta_i, \bar{\theta}^i)$.

The actions (2.2) and (2.3) are invariant under the following $\mathcal{N} = 2$ superconformal transformations of H , Υ and $\check{\Upsilon}$ [25]:

$$\zeta \delta H = - \left(\xi + \lambda^{++}(\zeta) \partial_\zeta \right) (\zeta H) - 2 \Sigma(\zeta) \zeta H , \quad (3.5a)$$

$$\delta \Upsilon^I = - \left(\xi + \lambda^{++}(\zeta) \partial_\zeta \right) \Upsilon^I , \quad \delta \check{\Upsilon}^{\bar{I}} = - \left(\xi + \lambda^{++}(\zeta) \partial_\zeta \right) \check{\Upsilon}^{\bar{I}} . \quad (3.5b)$$

Here ξ is a $\mathcal{N} = 2$ superconformal Killing vector,

$$\xi = \bar{\xi} = \xi^A(z) D_A = \xi^a(z) \partial_a + \xi^\alpha_i(z) D^\alpha_i + \bar{\xi}^{\dot{\alpha}}_{\dot{\alpha}}(z) \bar{D}^{\dot{\alpha}}_{\dot{\alpha}} , \quad (3.6)$$

with the master property

$$\bar{D}^{\dot{\alpha}}_i \Psi = 0 \quad \longrightarrow \quad \bar{D}^{\dot{\alpha}}_i (\xi \Psi) = 0 , \quad (3.7)$$

for any chiral superfield Ψ . The superconformal parameters $\lambda^{++}(\zeta)$ and $\Sigma(\zeta)$ appearing in (3.5a) and (3.5b) have the form

$$\lambda^{++}(\zeta) = \lambda^{11} \zeta^2 - 2\lambda^{12} \zeta + \lambda^{22} , \quad \Sigma(\zeta) = -\lambda^{11} \zeta + \lambda^{12} + \sigma + \bar{\sigma} \quad (3.8)$$

in terms of the descendants σ and λ^{ij} of ξ defined as

$$\begin{aligned} \sigma &= \frac{1}{4} \bar{D}^{\dot{\alpha}}_i \bar{\xi}^{\dot{\alpha}}_{\dot{\alpha}} , \quad \bar{D}^{\dot{\alpha}}_i \sigma = 0 \\ \lambda_j^i &= \frac{1}{2} \left(D^\alpha_i \xi^\alpha_j - \frac{1}{2} \delta_j^i D^\alpha_k \xi^\alpha_k \right) , \quad \lambda^{ij} = \lambda^{ji} , \quad \bar{\lambda}^{ij} = \lambda_{ij} . \end{aligned} \quad (3.9)$$

It should be remarked that these descendants originate as follows

$$[\xi , D^\alpha_i] = -(D^\alpha_i \xi^\beta_j) D^\beta_j = \omega^\beta_{\alpha\beta} D^\beta_i - \bar{\sigma} D^\alpha_i - \lambda_j^i D^\beta_j \quad \longrightarrow \quad \bar{D}^{\dot{\alpha}}_i \xi^\beta_j = 0 , \quad (3.10)$$

where

$$\omega_{\alpha\beta} = -\frac{1}{2} D^\alpha_i \xi^\beta_{(\alpha} \xi^\beta_{\beta)i} , \quad \bar{D}^{\dot{\alpha}}_i \omega_{\alpha\beta} = 0 . \quad (3.11)$$

See Refs. [31, 32, 33, 25] for more detail about superconformal transformations in $\mathcal{N} = 2$ superspace.

The superconformal transformation of $H(\zeta)$, eq. (3.5a), proves to be uniquely determined by the constraints obeyed by this multiplet, $D_\alpha(\zeta)H(\zeta) = \bar{D}_{\dot{\alpha}}(\zeta)H(\zeta) = 0$, and by its explicit dependence of ζ given by (2.4). Eq. (3.5b) means that $\Upsilon^I(\zeta)$ is a weight-zero arctic multiplet. The superconformal transformations of the weight- n arctic and antarctic multiplets are given by eqs. (A.2) and (A.3) respectively.

Consider the model (2.10) dual to (2.1). As discussed in section 2, it is $\mathcal{N} = 2$ superconformal invariance provided Υ^I and Ξ transform as the weight-zero and weight-one arctic multiplets, respectively, with the latter transformation law given by eq. (A.2) with $n = 1$.

It is of interest to analyze the superconformal properties of the auxiliary field equations (2.20). In complete analogy with the case $H = 1$ [34, 9], these equations imply that

$$\Omega_I(\zeta) := \zeta H \frac{\partial K(\Upsilon, \check{\Upsilon})}{\partial \Upsilon^I} \quad (3.12)$$

has no poles in ζ and therefore can be represented by a Taylor series

$$\Omega_I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Omega_{nI} . \quad (3.13)$$

This superfield becomes an arctic multiplet on the full mass shell when the equations of motion for Φ^I and Σ^I are imposed as well.

Let us promote the superfields H , Υ and $\check{\Upsilon}$ in (3.12) to $\mathcal{N} = 2$ projective superfields. Then, using the transformation laws (3.5a) and (3.5b), we observe that the composite (3.12) transforms as

$$\delta\Omega_I = -\left(\xi + \lambda^{++}(\zeta) \partial_\zeta\right)\Omega_I - 2\Sigma(\zeta)\Omega_I . \quad (3.14)$$

It is a simple exercise to check that this transformation law preserves the functional form of Ω_I given in (3.13). Therefore, the auxiliary field equations (2.20), or equivalently (3.13), are $\mathcal{N} = 2$ superconformal. On the full mass shell, eq. (3.14) tells us that Ω_I is a weight-two superconformal arctic multiplet.

We wish to convert the algebraic auxiliary field equations (2.20) into an equivalent second-order ordinary differential equation obeyed by $\Upsilon(\zeta)$, with ζ the evolution parameter. This is certainly possible in the case $H = 1$, as has been shown in [34, 35] for the Hermitian symmetric spaces, and in [25] for general Kähler spaces. In the case of an arbitrary $\mathcal{N} = 2$ tensor multiplet $H(\zeta)$, let us proceed to derive such an equation for a simplest Kähler potential.

3.2 Quadratic Kähler potential

The auxiliary field equations (2.20) can be explicitly solved in the case of a flat Kähler target space described by the potential⁵

$$\mathcal{K}(\Phi, \bar{\Phi}) = \Phi^\dagger \Phi = \delta_{I\bar{J}} \Phi^I \bar{\Phi}^{\bar{J}} . \quad (3.15)$$

Then, using eq. (2.9) we find

$$H \mathcal{K}(\Upsilon, \check{\Upsilon}) = H(\zeta) \check{\Upsilon}^{\bar{I}}(\zeta) \Upsilon^I(\zeta) = \frac{1}{2}(G + \mathbb{H}) \check{\Upsilon}^{\bar{I}}(\zeta) \Upsilon^I(\zeta) , \quad (3.16)$$

where

$$\Upsilon^I(\zeta) := \left(1 - \zeta \frac{2\bar{\varphi}}{G + \mathbb{H}}\right) \Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n^I . \quad (3.17)$$

⁵Our consideration can be trivially generalized to the case of an indefinite metric in (3.15), $\delta_{I\bar{J}} \rightarrow \eta_{I\bar{J}}$.

The new superfields, $\Upsilon^I(\zeta)$, are not arctic, for the components

$$\Upsilon_1^I = \Sigma^I - \frac{2\bar{\varphi}}{G + \mathbb{H}} \Phi^I \quad (3.18)$$

obey a modified linear constraint. The new auxiliary superfields $\Upsilon_2^I, \Upsilon_3^I, \dots$, can be immediately eliminated. As a result, the Lagrangian corresponding to the hypermultiplet action (2.3) with $K = \mathfrak{K}$ becomes

$$\mathfrak{L}_H = \frac{1}{2}(G + \mathbb{H}) \left\{ \bar{\Phi}^{\bar{I}} \Phi^I - \left(\bar{\Sigma}^{\bar{I}} - \frac{2\varphi}{G + \mathbb{H}} \bar{\Phi}^{\bar{I}} \right) \left(\Sigma^I - \frac{2\bar{\varphi}}{G + \mathbb{H}} \Phi^I \right) \right\}. \quad (3.19)$$

In terms of the Kähler potential $\mathfrak{K}(\Phi, \bar{\Phi})$, this Lagrangian can equivalently be rewritten in the form:

$$\mathfrak{L}_H = G \mathfrak{K} + \varphi \mathfrak{K}_I \Sigma^I + \bar{\varphi} \mathfrak{K}_{\bar{J}} \bar{\Sigma}^{\bar{J}} - \frac{1}{2}(G + \mathbb{H}) \mathfrak{K}_{I\bar{J}} \Sigma^I \bar{\Sigma}^{\bar{J}}. \quad (3.20)$$

Let $\Upsilon_*^I(\zeta)$ denote the unique solution to the *algebraic* auxiliary field equations under the initial conditions (2.21). It has the form

$$\Upsilon_*^I(\zeta) = \Phi^I + \frac{\zeta}{1 - \bar{\Lambda}\zeta} \Sigma^I, \quad (3.21)$$

where

$$\Lambda := \frac{2\varphi}{G + \mathbb{H}}. \quad (3.22)$$

It is further a solution to the following *differential* equation:

$$\frac{d^2 \Upsilon^I(\zeta)}{d\zeta^2} - 2 \frac{\bar{\Lambda}}{1 - \bar{\Lambda}\zeta} \frac{d\Upsilon^I(\zeta)}{d\zeta} = 0. \quad (3.23)$$

It is instructive to check that equation (3.23) is superconformal. Introduce the following superfield:

$$\Pi^I := \frac{d^2 \Upsilon^I(\zeta)}{d\zeta^2} - 2 \frac{\bar{\Lambda}}{1 - \bar{\Lambda}\zeta} \frac{d\Upsilon^I(\zeta)}{d\zeta}. \quad (3.24)$$

We are going to demonstrate that its superconformal transformation is

$$\delta \Pi^I = - \left(\xi + \lambda^{++} \partial_\zeta + 2(\partial_\zeta \lambda^{++}) \right) \Pi^I. \quad (3.25)$$

Using eq. (3.5b) gives

$$\frac{d}{d\zeta} \delta \Upsilon^I = - \left(\xi + \lambda^{++} \partial_\zeta + (\partial_\zeta \lambda^{++}) \right) \frac{d}{d\zeta} \Upsilon^I, \quad (3.26a)$$

$$\frac{d^2}{d\zeta^2} \delta \Upsilon^I = - \left(\xi + \lambda^{++} \partial_\zeta + 2(\partial_\zeta \lambda^{++}) \right) \frac{d^2}{d\zeta^2} \Upsilon^I - (\partial_\zeta^2 \lambda^{++}) \frac{d}{d\zeta} \Upsilon^I. \quad (3.26b)$$

Next, making use of eq. (3.5a) allows us to read off the superconformal transformations of the components of $H(\zeta)$:

$$\delta G = -\xi G - 2(\sigma + \bar{\sigma})G + 2\lambda^{22}\bar{\varphi} + 2\lambda^{11}\varphi , \quad (3.27a)$$

$$\delta\varphi = -\xi\varphi - 2(\sigma + \bar{\sigma})\varphi - \lambda^{22}G - 2\lambda^{12}\varphi , \quad (3.27b)$$

$$\delta\bar{\varphi} = -\xi\bar{\varphi} - 2(\sigma + \bar{\sigma})\bar{\varphi} - \lambda^{11}G + 2\lambda^{12}\varphi . \quad (3.27c)$$

These results immediately lead to

$$\delta\mathbb{H} = -\xi\mathbb{H} - 2(\sigma + \bar{\sigma})\mathbb{H} , \quad (3.28)$$

as well as to

$$\delta\Lambda = -\xi\Lambda - \lambda^{22} - 2\lambda^{12}\Lambda - \lambda^{11}\Lambda^2 , \quad \delta\bar{\Lambda} = -\xi\bar{\Lambda} - \lambda^{11} + 2\lambda^{12}\bar{\Lambda} - \lambda^{22}(\bar{\Lambda})^2 . \quad (3.29)$$

Making use of the results obtained, we check that

$$(\partial_\zeta^2 \lambda^{++}) + \left\{ \lambda^{++} \partial_\zeta + (\partial_\zeta \lambda^{++}) \right\} \frac{2\bar{\Lambda}}{1 - \bar{\Lambda}\zeta} + \frac{2(\delta\bar{\Lambda} + \xi\bar{\Lambda})}{(1 - \bar{\Lambda}\zeta)^2} = 0 . \quad (3.30)$$

The above identities indeed justify the superconformal transformation law (3.25), and hence the fact that the differential equation (3.23) is superconformal.

It should be pointed out that the hypermultiplet model (2.3) with Kähler potential (3.15) possesses a dual off-shell formulation obtained by dualizing each⁶ polar multiplet, Υ^I and $\check{\Upsilon}^{\bar{I}}$, into a real $\mathcal{O}(2)$ multiplet η^I , with $I = 1, \dots, n_H$.⁷ The dual formulation is described by the following $\mathcal{N} = 2$ superconformal action:

$$\mathfrak{S}_{H,\text{dual}} = - \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \frac{\eta^I \eta^I}{2H} . \quad (3.31)$$

Of the global $U(n_H)$ symmetry of the original hypermultiplet action, only its subgroup $O(n_H)$ is manifestly realized in the dual formulation, while the other symmetries emerge as duality transformations. In the same vein, of the $2n_H$ Abelian symmetries

$$\delta H(\zeta) = 0 , \quad \delta \Upsilon^I(\zeta) = c^I = \text{const} \quad (3.32)$$

of the original hypermultiplet model, only n_H (Peccei-Quinn-type) symmetries are manifestly realized in the dual formulation:

$$\delta H(\zeta) = 0 , \quad \delta \eta^I(\zeta) = H(\zeta) a^I , \quad a^I = \overline{a^{\bar{I}}} = \text{const} . \quad (3.33)$$

Modulo sign, the sigma model (3.31) with $n_H = 1$ is known to define the hyperkähler cone corresponding to the classical universal hypermultiplet [36, 15].

⁶The other option is to dualize only a subset of the n_H polar multiplets.

⁷As is well-known [28], this is possible only if the model possesses an isometry so that it does not depend on the phase of Υ .

3.3 Modified geodesic equation

If the Kähler space is not flat, the differential equation (3.23) is no-longer equivalent to the auxiliary field equations (2.20). Guided by the experience gained in the case $H = 1$ [34, 35, 37, 25] we should look for a generalization of eq. (3.23) of the form:

$$\mathbf{\Pi}^I = 0 , \quad (3.34)$$

where

$$\begin{aligned} \mathbf{\Pi}^I &:= \Pi^I + \Gamma^I_{JK} \left(\Upsilon(\zeta), \bar{\Phi} \right) \frac{d\Upsilon^J(\zeta)}{d\zeta} \frac{d\Upsilon^K(\zeta)}{d\zeta} + \dots \\ &\equiv \frac{d^2 \Upsilon^I(\zeta)}{d\zeta^2} - \frac{2\bar{\Lambda}}{1 - \bar{\Lambda}\zeta} \frac{d\Upsilon^I(\zeta)}{d\zeta} + \Gamma^I_{JK} \left(\Upsilon(\zeta), \bar{\Phi} \right) \frac{d\Upsilon^J(\zeta)}{d\zeta} \frac{d\Upsilon^K(\zeta)}{d\zeta} + \Delta \mathbf{\Pi}^I . \end{aligned} \quad (3.35)$$

Here the term containing the Christoffel symbol Γ^I_{JK} is required to ensure the correct transformation of $\mathbf{\Pi}^I$ under holomorphic reparametrizations (2.17). It can be argued that the last term in (3.35) must depend on the Kähler potential only via the corresponding Kähler metric, the Riemann tensor and its covariant derivatives. The superfield $\mathbf{\Pi}^I$ should be chosen such that

- (i) in the case when $H = 1$ and the Kähler manifold $\mathcal{M}_K^{2n_H}$ is Hermitian symmetric, eq. (3.34) should reduce to the geodesic equation (B.2);
- (ii) the $\mathcal{N} = 2$ superconformal transformation of $\mathbf{\Pi}^I$ should be

$$\delta \mathbf{\Pi}^I = - \left(\xi + \lambda^{++} \partial_\zeta + 2(\partial_\zeta \lambda^{++}) \right) \mathbf{\Pi}^I . \quad (3.36)$$

It turns out that the above requirements allow one, in principle, to reconstruct $\Delta \mathbf{\Pi}^I$ in (3.35) step by step in perturbation theory. As a first step, varying the right-hand side of (3.35) gives

$$\begin{aligned} \delta \mathbf{\Pi}^I &= - \left(\xi + \lambda^{++} \partial_\zeta + 2(\partial_\zeta \lambda^{++}) \right) \mathbf{\Pi}^I \\ &\quad - \lambda^{1\bar{1}} R_{J\bar{L}K}{}^I \left(\Upsilon(\zeta), \bar{\Phi} \right) \bar{\Sigma}^{\bar{L}} \frac{d\Upsilon^J(\zeta)}{d\zeta} \frac{d\Upsilon^K(\zeta)}{d\zeta} + \dots \end{aligned} \quad (3.37)$$

To derive this and some other relations below, one has to use the superconformal transformations of Φ , Σ and their conjugates:

$$\delta \Phi^I = -\xi \Phi^I - \lambda^{2\bar{2}} \Sigma^I , \quad (3.38a)$$

$$\delta \bar{\Phi}^{\bar{I}} = -\xi \bar{\Phi}^{\bar{I}} - \lambda^{1\bar{1}} \bar{\Sigma}^{\bar{I}} , \quad (3.38b)$$

$$\delta \Sigma^I = -\xi \Sigma^I + 2\lambda^{1\bar{2}} \Sigma^I - \lambda^{2\bar{2}} \Upsilon_2^I , \quad (3.38c)$$

$$\delta \bar{\Sigma}^{\bar{I}} = -\xi \bar{\Sigma}^{\bar{I}} - 2\lambda^{1\bar{2}} \bar{\Sigma}^{\bar{I}} - \lambda^{1\bar{1}} \bar{\Upsilon}_2^{\bar{I}} . \quad (3.38d)$$

These relations follow from (3.5b). In (3.38c) and (3.38d), Υ_2^I and its conjugate should be expressed in terms of the physical superfields Φ^I , Σ^I and their conjugates, in accordance with (3.34).

To cancel the variation in the second line of (3.37), it can be shown that the last term in (3.35) should have the form:

$$\Delta\Pi^I = -\frac{\bar{\Lambda}}{1+\Lambda\bar{\Lambda}} R_{J\bar{L}K}{}^I \left(\Upsilon(\zeta), \bar{\Phi} \right) \bar{\Sigma}^{\bar{L}} \frac{d\Upsilon^J(\zeta)}{d\zeta} \frac{d\Upsilon^K(\zeta)}{d\zeta} + O(R^2, \nabla R) . \quad (3.39)$$

Here $O(R^2, \nabla R)$ denotes terms of second and higher orders in the target space curvature, or terms containing covariant derivatives of the target space curvature.

3.4 Leading contributions to the hypermultiplet Lagrangian

The results obtained in the previous subsection allow us to restore several leading terms in the hypermultiplet Lagrangian $L_H(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ appearing in (2.22). Upon elimination of the auxiliary superfields, we find

$$\Upsilon_2^I = \bar{\Lambda}\Sigma^I - \frac{1}{2}\Gamma^I{}_{JK}\Sigma^J\Sigma^K + \frac{1}{2}\frac{\bar{\Lambda}}{1+\Lambda\bar{\Lambda}} R_{J\bar{L}K}{}^I \Sigma^J\Sigma^K\bar{\Sigma}^{\bar{L}} + O(\Sigma^4) , \quad (3.40)$$

where $O(\Sigma^4)$ denotes all the terms of fourth and higher powers in Σ and $\bar{\Sigma}$.

We now project the dynamical superfields to $\mathcal{N} = 1$ superspace and consider only the second Q-supersymmetry transformation [28, 8]:

$$\delta\varphi = \epsilon\overline{D}G , \quad (3.41a)$$

$$\delta G = -\epsilon D\varphi - \epsilon\overline{D}\bar{\varphi} , \quad (3.41b)$$

$$\delta\Phi^I = \epsilon\overline{D}\Sigma^I , \quad (3.41c)$$

$$\delta\Sigma^I = -\epsilon D\Phi^I + \epsilon\overline{D}\Upsilon_2^I , \quad (3.41d)$$

where Υ_2^I has to be expressed in terms of the dynamical superfields as in (3.40). These transformations follow from the relations (A.13a), (A.13b) and (A.16a), (A.16b) by setting $\rho^\alpha = \epsilon^\alpha = \text{const}$. Requiring the hypermultiplet action to possess this invariance, and also taking into account the fact that

$$L_H(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = GK + O(\Sigma) ,$$

one can show that

$$L_H = GK + \varphi K_I \Sigma^I + \bar{\varphi} K_{\bar{J}} \bar{\Sigma}^{\bar{J}} - \frac{1}{2}(G + \mathbb{H})g_{I\bar{J}}\Sigma^I\bar{\Sigma}^{\bar{J}} + O(\Sigma^4) . \quad (3.42)$$

It can readily be seen that the first three terms generate Kähler-invariant contributions to the action. The other terms in L_H prove to involve the Kähler potential only in the form of the Kähler metric $g_{I\bar{J}}$, the corresponding Riemann curvature $R_{I\bar{J}K\bar{L}}$ and its covariant derivatives.

4 Complex projective space

If the Kähler potential $K(\Phi, \bar{\Phi})$ in (2.3) corresponds to a generic Kähler manifold, it is not possible to obtain a closed-form expression for the modified geodesic equation (3.34,3.35) which is equivalent to the auxiliary field equations (2.20). In the non-superconformal case $H = 1$, this equation is known exactly for arbitrary Hermitian symmetric spaces [34, 35] and is given by eq. (B.2). Its extension to the superconformal case is quite nontrivial, due to the presence of an infinite number of curvature-dependent terms in (3.35). At the moment, we are not able to derive the equation (3.34,3.35) even for arbitrary Hermitian symmetric spaces. However, below we will work out explicitly one important example – the complex projective space $\mathcal{M} = \mathbb{C}P^{n_H} = SU(n_H + 1)/SU(n_H) \times U(1)$. We believe our consideration for $\mathbb{C}P^{n_H}$ can naturally be generalized to the case of arbitrary Hermitian symmetric spaces.

Using standard inhomogeneous coordinates for $\mathbb{C}P^{n_H}$, the Kähler potential⁸ and the metric are

$$K(\Phi, \bar{\Phi}) = r^2 \ln \left(1 + \frac{1}{r^2} \Phi^L \bar{\Phi}^{\bar{L}} \right), \quad g_{I\bar{J}}(\Phi, \bar{\Phi}) = \frac{r^2 \delta_{IJ}}{r^2 + \Phi^L \bar{\Phi}^{\bar{L}}} - \frac{r^2 \bar{\Phi}^{\bar{I}} \Phi^J}{(r^2 + \Phi^L \bar{\Phi}^{\bar{L}})^2}, \quad (4.1)$$

where $I, \bar{J} = 1, \dots, n_H$ and $r^2 = \text{const.}$ The Riemann curvature of $\mathbb{C}P^{n_H}$ is known to be

$$R_{I_1 \bar{J}_1 I_2 \bar{J}_2} := K_{I_1 \bar{J}_1 I_2 \bar{J}_2} - g_{M\bar{N}} \Gamma_{I_1 I_2}^M \bar{\Gamma}_{\bar{J}_1 \bar{J}_2}^{\bar{N}} = -\frac{1}{r^2} \left\{ g_{I_1 \bar{J}_1} g_{I_2 \bar{J}_2} + g_{I_1 \bar{J}_2} g_{I_2 \bar{J}_1} \right\}. \quad (4.2)$$

This implies

$$\Sigma^{I_1} \bar{\Sigma}^{\bar{J}_1} \Sigma^{I_2} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = -\frac{2}{r^2} g_{I_2 \bar{J}_2} \Sigma^{I_2} |\Sigma|^2, \quad (4.3)$$

where

$$|\Sigma|^2 := g_{I\bar{J}}(\Phi, \bar{\Phi}) \Sigma^I \bar{\Sigma}^{\bar{J}}. \quad (4.4)$$

As before, let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta; \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$ denote the unique solution of the auxiliary field equations (2.20) subject to the initial conditions (2.21). Then, the action (2.1) can

⁸Modulo an irrelevant constant, the Kähler potential $K(\Phi, \bar{\Phi})$ reduces to (3.15) in the limit $r \rightarrow \infty$.

be brought to the form (2.22), for some Lagrangian L_H . Instead of looking directly for $\Upsilon_*(\zeta)$, we will try to determine the Lagrangian L_H by making use of considerations based on extended supersymmetry, as a generalization of the approaches developed earlier in [38, 39] for the non-superconformal case $H = 1$.

For L_H we choose an ansatz of the form:

$$L_H(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = G K + \varphi K_I \Sigma^I + \bar{\varphi} K_{\bar{J}} \bar{\Sigma}^{\bar{J}} + L ,$$

$$L(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \equiv L(|\Sigma|^2) = \sum_{n=1}^{\infty} L_n |\Sigma|^{2n} , \quad L_n \equiv L_n(G, \varphi, \bar{\varphi}) , \quad (4.5)$$

where the first three terms in the expression for L_H agree with (3.42). The general structure of L given follows from the fact (4.4) is the only independent $U(n)$ -invariant that may be constructed in terms of Σ s and $\bar{\Sigma}$ s. At the moment, we only know that

$$L_1 = -\frac{1}{2}(G + \mathbb{H}) . \quad (4.6)$$

Our goal is to determine the other Taylor coefficients in (4.5), L_2, L_3, \dots , using extended supersymmetry.

Our strategy below will consist in trying to determine $L(|\Sigma|^2)$ by requiring the action

$$S_H = \int d^4x d^4\theta L_H(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \quad (4.7)$$

to be invariant under the second Q-supersymmetry transformation (3.41a – 3.41d), with Υ_2^I currently an unknown function of the physical superfields which has to be determined. We choose the following ansatz for Υ_2^I :

$$\Upsilon_2^I = -\frac{1}{2}\Gamma_{JK}^I \Sigma^J \Sigma^K + \Sigma^I \sum_{n=0}^{\infty} c_n |\Sigma|^{2n} , \quad c_n \equiv c_n(G, \varphi, \bar{\varphi}) . \quad (4.8)$$

At the moment, we only know that

$$c_0 = \bar{\Lambda} = \frac{2\bar{\varphi}}{G + \mathbb{H}} , \quad (4.9)$$

in accordance with (3.40). Our goal is to determine the other Taylor coefficients in (4.8), c_1, c_2, \dots , using extended supersymmetry.

Let us vary the action with respect to the second Q-supersymmetry transformation (3.41a – 3.41d), keep the $\bar{\epsilon}$ -dependent terms only, and analyze what conditions are necessary for S_H to be invariant. The variation $\delta_{\bar{\epsilon}} S_H$ involves two types of terms containing

even and odd powers of Σ s and $\bar{\Sigma}$ s respectively. The requirement that all even terms vanish can be shown to be equivalent to the following two conditions:

$$\sum_{k=0}^{n-1} (n-k)c_k L_{n-k} = 0, \quad n \geq 2 \quad (4.10)$$

$$\delta_{\bar{\epsilon}} L_n + \sum_{k=1}^n k L_k \bar{\epsilon} \bar{D} c_{n-k} - \frac{1}{n} \sum_{k=1}^n k(n-k) \bar{\epsilon} \bar{D} (L_k c_{n-k}) = 0. \quad (4.11)$$

The requirement that all odd terms vanish can be shown to be equivalent to the following condition:

$$(n+1)L_{n+1} = \frac{n}{r^2} L_n - \varphi c_n, \quad n \geq 1. \quad (4.12)$$

Before continuing the general analysis, let us briefly pause and make a simple check of equation (4.11), by considering the choice $n = 1$, that is

$$\delta_{\bar{\epsilon}} L_1 + L_1 \bar{\epsilon} \bar{D} c_0 = 0. \quad (4.13)$$

Since

$$\delta_{\bar{\epsilon}} (G + \mathbb{H}) = -(G + \mathbb{H}) \bar{\epsilon} \bar{D} \bar{\Lambda}, \quad (4.14)$$

the relations (4.6) and (4.9) imply that (4.13) is identically satisfied.

Using the relations (4.10) and (4.12), we can derive a recursion relation to determine the coefficients L_n . It is

$$L_n = \frac{1}{n\mathbb{H}} \left\{ \sum_{k=1}^{n-2} (n-k)(k+1) L_{n-k} L_{k+1} - \frac{1}{r^2} \sum_{k=1}^{n-1} (n-k)k L_{n-k} L_k \right\}, \quad n \geq 3. \quad (4.15)$$

For $n = 2$, only the second term on the right contributes, hence

$$L_2 = -\frac{1}{2r^2\mathbb{H}} (L_1)^2 = -\frac{1}{8r^2} \frac{(G + \mathbb{H})^2}{\mathbb{H}}. \quad (4.16)$$

Making use of eq. (4.15) allows one to obtain an algebraic equation obeyed by

$$L'(x) = \sum_{n=1}^{\infty} n L_n x^{n-1}.$$

The equation is

$$(1 - x/r^2) [L'(x)]^2 + GL'(x) - \frac{1}{4} (\mathbb{H}^2 - G^2) = 0. \quad (4.17)$$

We have to choose the following solution of the quadratic equation obtained:

$$L'(x) = -\frac{1}{2} \frac{G}{1-x/r^2} - \frac{1}{2} \frac{\sqrt{G^2 + (\mathbb{H}^2 - G^2)(1-x/r^2)}}{1-x/r^2}, \quad (4.18)$$

for it possesses the right functional form in the limit $\mathbb{H} \rightarrow G$. Now, the problem of computing $L(x)$ amounts to doing an ordinary integral. The result is as follows:

$$L(x) = -r^2 \left\{ \mathbb{H} - G \ln(G + \mathbb{H}) \right\} + r^2 \sqrt{G^2 + (\mathbb{H}^2 - G^2)(1-|\Sigma|^2/r^2)} \\ + r^2 G \ln \frac{1-|\Sigma|^2/r^2}{\sqrt{G^2 + (\mathbb{H}^2 - G^2)(1-|\Sigma|^2/r^2)} + G}. \quad (4.19)$$

It can be seen that

$$\lim_{\mathbb{H} \rightarrow G} L(x) = G r^2 \ln \left(1 - \frac{x}{r^2} \right) \quad (4.20)$$

which agrees with [35, 40, 37, 38].

Using the relations (4.12), we can now compute all the coefficients c_n , and hence the function $c(x)$ appearing in (4.8). The latter is

$$c(x) := \sum_{n=0}^{\infty} c_n x^n = 2\bar{\varphi} \frac{1-x/r^2}{\sqrt{G^2 + (\mathbb{H}^2 - G^2)(1-x/r^2)} + G}. \quad (4.21)$$

So far we have determined $L(x)$ and $c(x)$ by using the relations (4.10) and (4.12). It still remains to be checked that eq. (4.11) is also satisfied. Instead of enjoying such an exercise, we choose a different course.

In accordance with (4.19), upon elimination of the auxiliary superfields, the hypermultiplet Lagrangian is

$$L_H = G K(\Phi, \bar{\Phi}) + \varphi K_I(\Phi, \bar{\Phi}) \Sigma^I + \bar{\varphi} K_{\bar{J}}(\Phi, \bar{\Phi}) \bar{\Sigma}^{\bar{J}} - r^2 \left\{ \mathbb{H} - G \ln(G + \mathbb{H}) \right\} \\ + r^2 \left\{ G \ln \frac{1-|\Sigma|^2/r^2}{\sqrt{G^2 + 4\bar{\varphi}\varphi(1-|\Sigma|^2/r^2)} + G} + \sqrt{G^2 + 4\bar{\varphi}\varphi(1-|\Sigma|^2/r^2)} \right\}. \quad (4.22)$$

Consider the second Q-supersymmetry transformation (3.41a–3.41d), where

$$\Upsilon_2^I = -\frac{1}{2} \Gamma_{JK}^I \Sigma^J \Sigma^K + 2\Sigma^I \bar{\varphi} \frac{1-|\Sigma|^2/r^2}{\sqrt{G^2 + 4\bar{\varphi}\varphi(1-|\Sigma|^2/r^2)} + G}. \quad (4.23)$$

It is an instructive, albeit time consuming, exercise to check explicitly that the action (4.7), with L_H given by (4.22), is invariant under this transformation. This implies that all of the equations (4.11) are identically satisfied.

One can readily check that the action (4.7) generated by the Lagrangian L_H , eq. (4.22), is invariant under the $\mathcal{N} = 1$ superconformal transformation (A.12), (A.15) and the shadow chiral rotation (A.14), (A.17), where n should be set to zero for both transformations. We leave it as an exercise for the reader to check that the action is also invariant under arbitrary extended superconformal transformations (A.13a), (A.13b) with $n = 0$ and (A.16a), (A.16b).

The hypermultiplet model (4.7), with L_H given by (4.22), possesses a dual formulation obtained by dualizing the complex linear tangent variables Σ^I and their conjugates $\bar{\Sigma}^{\bar{I}}$ into chiral one-forms Ψ_I and their conjugates $\bar{\Psi}_{\bar{I}}$, $\bar{D}_{\dot{\alpha}} \Psi_I = 0$. As usual, one first replaces the action with a first order one,

$$S = \int d^4x d^4\theta \left\{ L_H(G, \varphi, \bar{\varphi}, \Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Sigma^I \Psi_I + \bar{\Sigma}^{\bar{J}} \bar{\Psi}_{\bar{J}} \right\}, \quad (4.24)$$

where Σ^I and $\bar{\Sigma}^{\bar{J}}$ are chosen to be complex unconstrained. Next, one eliminates these superfields with the aid of their algebraic equations of motions, ending up with the dual Lagrangian:

$$L_H^{(\text{dual})} = G K(\Phi, \bar{\Phi}) - r^2 \left\{ \mathbb{H} - G \ln(G + \mathbb{H}) \right\} + r^2 \left\{ \sqrt{\mathbb{H}^2 + 4|\Psi + \varphi \nabla K|^2 / r^2} - G \ln \left(\sqrt{\mathbb{H}^2 + 4|\Psi + \varphi \nabla K|^2 / r^2} + G \right) \right\}, \quad (4.25)$$

where

$$|\Psi + \varphi \nabla K|^2 := g^{I\bar{J}} \left(\Psi_I + \varphi K_I(\Phi, \bar{\Phi}) \right) \left(\bar{\Psi}_{\bar{J}} + \bar{\varphi} K_{\bar{J}}(\Phi, \bar{\Phi}) \right). \quad (4.26)$$

Under the Kähler transformation (2.13), the chiral one-form Ψ_I changes as

$$\Psi_I \longrightarrow \Psi_I - \varphi F_I(\Phi), \quad (4.27)$$

and this transformation is clearly consistent with the chirality of Ψ_I . In the limit $G = 1$ and $\varphi = 0$, the Lagrangian (4.25) reduces to the hyperkähler potential for the cotangent bundle of $\mathbb{C}P^{n_H}$ [41], see [28, 40] and references therein for alternative supersymmetric techniques to derive the Calabi metric.

To conclude this section, we should mention that the above consideration for the complex projective space $\mathcal{M} = SU(n_H + 1) / SU(n_H) \times U(1)$ can be immediately generalized to the non-compact space $SU(n_H, 1) / SU(n_H) \times U(1)$ characterized by the Kähler potential

$$K(\Phi, \bar{\Phi}) = -r^2 \ln \left(1 - \frac{1}{r^2} \Phi^L \bar{\Phi}^{\bar{L}} \right). \quad (4.28)$$

This generalization amounts to replacing everywhere $r^2 \longrightarrow -r^2$.

5 Discussion

In section 4, we studied the dynamical system (2.1) for the case when the Kähler potential has the form (4.1) and corresponds to $\mathbb{C}P^{n_H}$. Some aspects of this theory are more transparent within its dual formulation (2.10) in which the action, modulo a trivial rescaling of Υ^I , is

$$S_{\text{dual}} = \kappa \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \check{\Xi} \Xi (1 + \Upsilon^I \check{\Upsilon}^{\bar{I}})^m, \quad m := \frac{r^2}{\kappa}. \quad (5.1)$$

This formulation is useful to see that the parameter m should be an integer (compare with [3]). It is sufficient to consider the case of $\mathbb{C}P^1$. Then Υ is the inhomogeneous complex coordinate in one of the two standard charts for $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$, say in the chart \mathbb{C} . Let Υ' be the complex coordinate in the second chart, $\mathbb{C}^* \cup \{\infty\}$, with $\mathbb{C}^* := \mathbb{C} - \{0\}$, such that the transition function is $\Upsilon' = 1/\Upsilon$. Of course, the action (5.1) for $\mathbb{C}P^1$ should be well-defined in both charts. In the second chart, it reads

$$S_{\text{dual}} = \kappa \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta \check{\Xi}' \Xi' (1 + \Upsilon' \check{\Upsilon}')^m, \quad \Xi' = \Xi \Upsilon^m. \quad (5.2)$$

In order for the compensator Ξ' to be well-defined on \mathbb{C}^* , the parameter m should be an integer. In the general case of $\mathbb{C}P^{n_H}$, similar arguments show that the variables Υ^I and Ξ parametrize a holomorphic line bundle over $\mathbb{C}P^{n_H}$.

More generally, the arctic variables Υ^I and Ξ in the model (2.10) should parametrize a holomorphic line bundle over a Kähler-Hodge manifold $\mathcal{M}_K^{2n_H}$ with Kähler potential

$$\mathbb{K}(\Phi, \bar{\Phi}) = \frac{1}{\kappa} K(\Phi, \bar{\Phi}), \quad (5.3)$$

in order for the action to be well-defined. To justify this claim, it suffices to reiterate the discussion of Kähler-Hodge geometry given in [42] (see also [43] for a recent review). Let $\omega = i\partial\bar{\partial}\mathbb{K}$ be the Kähler two-form of $\mathcal{M}_K^{2n_H}$. The Kähler manifold is Hodge if $\omega/2\pi \in H^2(\mathcal{M}_K^{2n_H}, \mathbb{Z})$, where $H^2(\mathcal{M}_K^{2n_H}, \mathbb{Z})$ denotes the second cohomology group of $\mathcal{M}_K^{2n_H}$ with integer coefficients. Then, one can associate with ω a holomorphic line bundle with connection for which ω is the field strength. The Kähler potential \mathbb{K} can be chosen such that $h := e^{\mathbb{K}}$ is a Hermitian fiber metric on the line bundle, $\|\chi\|^2 = h \chi \bar{\chi}$. Given a nowhere vanishing *local* section χ of the line bundle, the Kähler potential can be given, in accordance with [42], as $\mathbb{K} = \ln \|\chi\|^2$. This geometric picture extends to the $\mathcal{N} = 2$ supersymmetric case by replacing $\Phi^I \rightarrow \Upsilon^I$ and $\chi \rightarrow \Xi$. The crucial point is that the action (2.10) is globally well-defined in spite of the fact that the Lagrangian is given in terms of local data.

Our discussion above shows that the dual formulation (2.10) with arctic compensator requires Kähler-Hodge geometry. An interesting question is: Can we see the same geometry within the formulation (2.1) with tensor compensator? The answer is “Yes” provided the action (2.1) is rewritten in the following equivalent form:

$$S[H(\zeta), \Upsilon(\zeta)] = \kappa \oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta H \ln \frac{e^{\mathbb{K}(\Upsilon, \check{\Upsilon})} \Xi \check{\Xi}}{H} . \quad (5.4)$$

Here $\Xi(\zeta)$ is a weight-one arctic multiplet, and $\check{\Xi}(\zeta)$ its smile-conjugate. These multiplets are purely gauge degrees of freedom, for (5.4) is invariant under gauge transformations of the form:

$$\Xi \longrightarrow \Xi' = e^\rho \Xi , \quad (5.5)$$

with ρ an arbitrary weight-zero arctic multiplet. The gauge invariance follows from the identity

$$\oint \frac{d\zeta}{2\pi i \zeta} \int d^4x d^4\theta H \rho = 0 . \quad (5.6)$$

Unlike the original action (2.1), its reformulation (5.4) is manifestly $\mathcal{N} = 2$ superconformal.⁹ The arctic variables Υ^I and Ξ in (5.4) parametrize the holomorphic line bundle over $\mathcal{M}_K^{2n_H}$ introduced earlier.

It should be pointed out that no quantization condition occurs in the case of non-conformal $\mathcal{N} = 2$ sigma model (2.19). The Kähler potential in (2.19) is required to be real analytic but is otherwise arbitrary. The point is that the component Lagrangian can be defined as (compare with [44])

$$L_{\text{component}} = \frac{1}{16} D^\alpha \bar{D}^2 D_\alpha \oint \frac{d\zeta}{2\pi i \zeta} K(\Upsilon, \check{\Upsilon}) = \frac{1}{16} \bar{D}_\alpha D^2 \bar{D}^\alpha \oint \frac{d\zeta}{2\pi i \zeta} K(\Upsilon, \check{\Upsilon}) , \quad (5.7)$$

and it is manifestly invariant under Kähler transformation (2.14).

Let us return to the case $\mathcal{M}_K^{2n_H} = \mathbb{C}P^{n_H}$ discussed at the beginning of this section. As follows from (5.1), the choice

$$r^2 = \kappa \quad (5.8)$$

corresponds to a free theory, and this property should also be seen within the original model (2.1). Indeed, for this particular choice of parameters the fourth term in the expression (4.22) for L_H (or the second term in the expression (4.25) for the dual Lagrangian

⁹The action (5.4) is the rigid superspace version of a locally supersymmetric action introduced in [21].

$L_H^{(\text{dual})}$) cancels against κL_T , with L_T the tensor multiplet Lagrangian, eq. (2.7). Now, in the theory with Lagrangian

$$r^2 L_T + L_H^{(\text{dual})} = r^2 G \left\{ \ln \left(1 + \frac{1}{r^2} \Phi^\dagger \Phi \right) - \ln \left(\sqrt{\mathbb{H}^2 + 4|\Psi + \varphi \nabla K|^2 / r^2} + G \right) \right\} \\ + r^2 \sqrt{\mathbb{H}^2 + 4|\Psi + \varphi \nabla K|^2 / r^2} , \quad (5.9)$$

one can explicitly dualize the real linear superfield G into a chiral scalar χ and its conjugate. Modulo a field redefinition, the action obtained describes $n_H + 1$ free hypermultiplets. Thus, in spite of the fact that the above Lagrangian is nonlinear, it generates free dynamics. This is analogous to the situation with the improved tensor multiplet (2.7) described in detail in [28].

In our analysis of the modified geodesic equation in section 3, we started with the simplest case of a flat Kähler target space characterized by the Kähler potential (3.15). This case is actually interesting on its own. As mentioned at the end of subsection 3.2, the polar multiplets Υ^I and $\check{\Upsilon}^{\bar{I}}$ can be dualized into real $\mathcal{O}(2)$ multiplets η^I such that the resulting hypermultiplet action is given by (3.31). This action for $n_H = 1$ provides the projective superspace description [15] for the classical universal hypermultiplet [45]. Combining this action with the tensor multiplet sector in (2.1), we obtain a theory of two tensor multiplets with Lagrangian

$$\mathfrak{L}_{\text{UHM}} = -\kappa H \ln H - \frac{1}{2} \frac{\eta^2}{H} \quad (5.10)$$

which is (modulo sign) the projective superspace description [46] (see also [47]) of the one-loop corrected universal hypermultiplet [48].

There are various interesting problems that can be addressed building on the results of this paper. In particular, it is of interest to extend the analysis for the complex projective space given in section 4 to the case of arbitrary Hermitian symmetric spaces. This should include the derivation of closed-form expressions for the modified geodesic equation and the hypermultiplet action S_H . Such expressions are actually known if we set $\varphi = 0$ and keep only the real linear superfield G of the tensor multiplet $H(\zeta)$. Then, the auxiliary field equations (2.20) reduce to those corresponding to the non-superconformal model (2.19). The latter are equivalent, if \mathcal{M} is Hermitian symmetric, to the geodesic equation (B.2). As to the hypermultiplet action S_H , it is obtained by inserting G into the integrand of (B.3). The real challenge, however, is to extend these simple results, corresponding to the special case $H = G$, to the general tensor multiplet (2.4). In the supergravity context, the local $\text{SU}(2)$ invariance allows one to choose the gauge $H = G$, see [49] for a related

discussion. But for the rigid superconformal sigma models under consideration, we have at our disposal only rigid $SU(2)$ transformations that cannot be used to choose the gauge $\varphi = 0$ (compare with [19] where such a gauge condition was nevertheless employed).

In conclusion, we mention that our results can be used to study the dynamics of a family of nonlinear sigma models in $\mathcal{N} = 2$ anti-de Sitter superspace proposed in [50]. Such sigma models are described by the action (2.3) in which H is a background tensor multiplet containing all the information about the anti-de Sitter supergeometry.

Acknowledgements:

UL and RvU acknowledge the hospitality of the School of Physics at UWA where this work was initiated. We are grateful to Simon Tyler for assistance with Mathematica. The research of UL was supported by VR grant 621-2006-3365 and that of RvU by Czech ministry of education contract No. MSM0021622409.

A $\mathcal{N} = 2$ superconformal transformations and their realization in $\mathcal{N} = 1$ superspace

General 4D $\mathcal{N} = 2$ superconformal projective multiplets and their superconformal couplings were described in detail¹⁰ in [25], building on the earlier equivalent results in five dimensions [51]. Here we list the $\mathcal{N} = 2$ superconformal transformations of several off-shell supermultiplets and their realization in $\mathcal{N} = 1$ superspace following [25].

Let $\Upsilon^{[n]}(\zeta)$ be an arctic weight- n multiplet,

$$\Upsilon^{[n]}(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n . \quad (\text{A.1})$$

Its $\mathcal{N} = 2$ superconformal transformation is as follows:

$$\delta \Upsilon^{[n]} = - \left(\xi + \lambda^{++}(\zeta) \partial_{\zeta} \right) \Upsilon^{[n]} - n \Sigma(\zeta) \Upsilon^{[n]} . \quad (\text{A.2})$$

The smile-conjugate of $\Upsilon^{[n]}(\zeta)$ is the weight- n antarctic multiplet denoted as $\check{\Upsilon}^{[n]}(\zeta)$. Its superconformal transformation is

$$\delta \check{\Upsilon}^{[n]} = - \frac{1}{\zeta^n} \left(\xi + \lambda^{++}(\zeta) \partial_{\zeta} \right) (\zeta^n \check{\Upsilon}^{[n]}) - n \Sigma(\zeta) \check{\Upsilon}^{[n]} . \quad (\text{A.3})$$

¹⁰Superconformal $\mathcal{O}(n)$ multiplets and their couplings were also studied in [52, 53], however their analysis was restricted to deriving the conditions for invariance under the $SU(2)$ transformations and dilations. Unlike the more general analysis presented in [25], no results were given in [52, 53] for the most interesting superconformal projective multiplets – the polar and tropical multiplets.

In the case $n = 0$, these transformations reduce to (3.5b).

As shown in [25], the transformation of $\mathcal{N} = 2$ supermultiplets associated with the $\mathcal{N} = 2$ superconformal Killing vector ξ generates three types of transformations at the level of $\mathcal{N} = 1$ superfields. They are:

1. An arbitrary $\mathcal{N} = 1$ superconformal transformation generated by

$$\xi = \bar{\xi} = \xi^a \partial_a + \xi^\alpha D_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \quad (\text{A.4})$$

such that

$$[\xi, D_\alpha] = \omega_\alpha{}^\beta D_\beta + (\sigma - 2\bar{\sigma}) D_\alpha, \quad (\text{A.5})$$

see [24] for more detail. The components of ξ and their descendants $\omega_\alpha{}^\beta$ and σ correspond to the following choice of the $\mathcal{N} = 2$ parameters:

$$\xi| = \xi, \quad \omega_\alpha{}^\beta| = \omega_\alpha{}^\beta, \quad \sigma| = \sigma, \quad \lambda_{\underline{1}}^{\underline{1}}| = \bar{\sigma} - \sigma, \quad \lambda_{\underline{2}}^{\underline{1}}| = 0. \quad (\text{A.6})$$

2. An extended superconformal transformation generated by

$$\begin{aligned} \xi| &= \rho^\alpha D_\alpha^2 + \bar{\rho}_{\dot{\alpha}} \bar{D}_{\underline{2}}^{\dot{\alpha}}, & \xi_{\underline{2}}^\alpha| &= \rho^\alpha, \\ \omega_\alpha{}^\beta| &= \sigma| = \lambda_{\underline{1}}^{\underline{1}}| = 0, & \lambda_{\underline{2}}^{\underline{1}}| &= \lambda^{\underline{1}\underline{1}}| = -\frac{1}{2} D^\alpha \rho_\alpha. \end{aligned} \quad (\text{A.7})$$

3. A shadow chiral rotation. This is a phase transformation of $\theta_{\underline{2}}^\alpha$ only, with $\theta_{\underline{1}}^\alpha$ kept unchanged, and it corresponds to the choice

$$\xi| = 0, \quad \omega_\alpha{}^\beta| = \lambda_{\underline{2}}^{\underline{1}}| = 0, \quad \sigma| = \lambda_{\underline{1}}^{\underline{1}}| = -\bar{\sigma}| = -\frac{i}{2} \alpha. \quad (\text{A.8})$$

The spinor parameter ρ^α in (A.7) can be shown to obey the equations

$$\bar{D}_{\dot{\alpha}} \rho^\beta = 0, \quad D^{(\alpha} \rho^{\beta)} = 0, \quad (\text{A.9})$$

and the latter imply

$$\partial^{\dot{\alpha}(\alpha} \rho^{\beta)} = D^2 \rho^\beta = 0. \quad (\text{A.10})$$

There are several ordinary (component) transformations generated by the chiral spinor ρ^α in (A.7): (i) second Q-supersymmetry transformation (ϵ^α); (ii) off-diagonal SU(2)-transformation ($f = \lambda^{\underline{1}\underline{1}}|_{\theta=0}$); (iii) second S-supersymmetry transformation ($\bar{\eta}_{\dot{\alpha}}$). They emerge as follows:

$$\rho^\alpha(x_{(+)}, \theta) = \epsilon^\alpha + f \theta^\alpha - i \bar{\eta}_{\dot{\alpha}} x_{(+) }^{\dot{\alpha}\alpha}, \quad (\text{A.11})$$

with $x_{(+)}^a$ the chiral extension of x^a .

Consider the arctic weight- n multiplet $\Upsilon^{[n]}(\zeta)$. Its $\mathcal{N} = 2$ superconformal transformation law (A.2) generates the following $\mathcal{N} = 1$ variations of the component superfields:

1. the $\mathcal{N} = 1$ superconformal transformation

$$\delta\Upsilon_k = -\xi\Upsilon_k - 2k(\bar{\sigma} - \sigma)\Upsilon_k - 2n\sigma\Upsilon_k ; \quad (\text{A.12})$$

2. the extended superconformal transformation

$$\delta\Upsilon_0 = \bar{\rho}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\Upsilon_1 + \frac{1}{2}(\bar{D}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}})\Upsilon_1 , \quad (\text{A.13a})$$

$$\delta\Upsilon_1 = -\rho^{\alpha}D_{\alpha}\Upsilon_0 + \bar{D}_{\dot{\alpha}}(\bar{\rho}^{\dot{\alpha}}\Upsilon_2) - \frac{n}{2}(D^{\alpha}\rho_{\alpha})\Upsilon_0 , \quad (\text{A.13b})$$

$$\begin{aligned} \delta\Upsilon_k &= -\rho^{\alpha}D_{\alpha}\Upsilon_{k-1} + \bar{\rho}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\Upsilon_{k+1} \\ &\quad + \frac{1}{2}(k-n-1)(D^{\alpha}\rho_{\alpha})\Upsilon_{k-1} + \frac{1}{2}(k+1)(\bar{D}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}})\Upsilon_{k+1} , \quad k > 1 ; \end{aligned} \quad (\text{A.13c})$$

3. the shadow chiral rotation

$$\delta\Upsilon_k = i\alpha(k - \frac{n}{2})\Upsilon_k . \quad (\text{A.14})$$

Choosing $n = 0$ in the above relations, one obtains the transformations of the dynamical superfields $\Phi := \Upsilon_0$ and $\Sigma := \Upsilon_1$ of the weight-zero arctic multiplet Υ .

Consider the tensor multiplet $H(\zeta)$, eq. (2.4). Its $\mathcal{N} = 2$ superconformal transformation law (3.5a) generates the following $\mathcal{N} = 1$ transformations of the component superfields:

1. the $\mathcal{N} = 1$ superconformal transformation

$$\delta\varphi = -\xi\varphi - 4\sigma\varphi , \quad \delta G = -\xi G - 2(\sigma + \bar{\sigma})G ; \quad (\text{A.15})$$

2. the extended superconformal transformation

$$\delta\varphi = \bar{\rho}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}G + \frac{1}{2}(\bar{D}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}})G , \quad (\text{A.16a})$$

$$\delta G = -\rho^{\alpha}D_{\alpha}\varphi - \bar{\rho}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\bar{\varphi} - \left((D^{\alpha}\rho_{\alpha})\varphi + (\bar{D}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}})\bar{\varphi} \right) ; \quad (\text{A.16b})$$

3. the shadow chiral rotation

$$\delta\varphi = -i\alpha\varphi , \quad \delta G = 0 . \quad (\text{A.17})$$

B Supersymmetric sigma models on tangent bundles of Hermitian symmetric spaces

This appendix contains a summary of several results obtained in a series of papers [34, 35, 40, 37, 38, 39] devoted to the study of $\mathcal{N} = 2$ supersymmetric sigma models of

the form (2.19), where $K(\Phi, \bar{\Phi})$ is the Kähler potential of a Hermitian symmetric space, and therefore the corresponding curvature tensor is covariantly constant,

$$\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = 0 . \quad (\text{B.1})$$

In such a model, the auxiliary field equations are equivalent to the geodesic equation with complex evolution parameter [34, 35]

$$\frac{d^2 \Upsilon^I(\zeta)}{d\zeta^2} + \Gamma^I_{JK} \left(\Upsilon(\zeta), \bar{\Phi} \right) \frac{d\Upsilon^J(\zeta)}{d\zeta} \frac{d\Upsilon^K(\zeta)}{d\zeta} = 0 . \quad (\text{B.2})$$

Upon elimination of the auxiliary superfields, the action (2.19) can be shown to take the form [39]:

$$S[\Upsilon_*(\zeta)] = \int d^4x d^4\theta \left\{ K(\Phi, \bar{\Phi}) - \frac{1}{2} \Sigma^T \mathbf{g} \frac{\ln(\mathbb{1} + \mathbf{R}_{\Sigma, \bar{\Sigma}})}{\mathbf{R}_{\Sigma, \bar{\Sigma}}} \Sigma \right\} , \quad \Sigma := \begin{pmatrix} \Sigma^I \\ \bar{\Sigma}^{\bar{I}} \end{pmatrix} , \quad (\text{B.3})$$

where

$$\mathbf{R}_{\Sigma, \bar{\Sigma}} := \begin{pmatrix} 0 & (R_\Sigma)^I_{\bar{J}} \\ (R_{\bar{\Sigma}})^{\bar{I}}_J & 0 \end{pmatrix} , \quad (R_\Sigma)^I_{\bar{J}} := \frac{1}{2} R_K{}^I{}_{L\bar{J}} \Sigma^K \Sigma^{\bar{L}} , \quad (R_{\bar{\Sigma}})^{\bar{I}}_J := \overline{(R_\Sigma)^I_{\bar{J}}} , \quad (\text{B.4})$$

and

$$\mathbf{g} := \begin{pmatrix} 0 & g_{I\bar{J}} \\ g_{\bar{I}J} & 0 \end{pmatrix} . \quad (\text{B.5})$$

Here $\Upsilon_*(\zeta)$ denotes the unique solution of equation (B.2) under the initial conditions (2.21). A different universal representation for the action $S[\Upsilon_*(\zeta)]$ can be found in [38].

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